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Abstract. We consider the randomly forced invariant manifolds of nonlinear dynamic systems. To study the dispersion of random states near the general deterministic attractors, we discuss two approaches. The first approach is based on the approximation of the quasipotential, and the second one uses the linear extension systems. A new semi-analytical method based on the stochastic sensitivity functions is suggested. The corresponding mathematical theory is shortly presented. Constructive applications of this theory to the analysis of equilibria and oscillatory regimes are given.

INTRODUCTION

Many nonlinear dynamical phenomena are related to the chain of bifurcations: a stationary regime – periodic regime – quasiperiodic regime – chaos [1]. Each regime is associated with the specific type of attractors (*e.g.*, equilibrium, limit cycle, torus, strange attractor). An invariant manifold can serve as a convenient mathematical model for the elaboration of the general methods for the analysis of these attractors. An important direction of the scientific research is connected with the study of the impact of random disturbances [2, 6].

In the present paper, we study an influence of the random noise on the general stochastic systems with the smooth compact invariant manifolds. A technique of the stochastic sensitivity functions defined on the manifolds is developed. Here, the quadratic approximation of the quasipotential is used. A new mathematical analysis based on the stochastic linear extension systems is elaborated. The present paper generalizes the results on the stochastic sensitivity of equilibria, limit cycles, and tori [7, 8, 9, 10].

EXPONENTIAL STABILITY OF INVARIANT MANIFOLDS

Consider a deterministic nonlinear system

$$dx \;=\; f(x)\,dt, \tag{1}$$

where x is an n-vector, $f(x)$ is a sufficiently smooth n-vector-function. It is assumed that the system (1) has a smooth compact invariant manifold $\mathcal{M}$ [11, 12, 13].

In the neighborhood U of the manifold $\mathcal{M}$, consider a function $\gamma(x)$, where $\gamma(x)$ is a point of the manifold $\mathcal{M}$ that is nearest to the point x. Denote by $\Delta(x) = x - \gamma(x)$ a vector of the deviation of the point x from the manifold $\mathcal{M}$. We assume that the neighborhood U is invariant for system (1).

For any $x \in \mathcal{M}$, denote by T_x the tangent subspace to $\mathcal{M}$ at the point x. Denote by N_x the orthogonal complement to T_x, and by P_x the operator of the orthogonal projection onto the subspace N_x.

Definition 1. *The invariant manifold $\mathcal{M}$ is called exponentially stable (E-stable) for the system (1) in U if there exist constants $K > 0$, $l > 0$ such that*

$$\|\Delta(x(t))\|^2 \;\le\; Ke^{-lt}\|\Delta(x_0)\|^2,$$

where $x(t)$ is a solution of system (1) with the initial condition $x(0) = x_0 \in U$.
The criterion of the exponential stability of invariant manifolds was given in [14].

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Consider a space Σ of the symmetrical $n \times n$-matrix functions $V(x)$ defined and sufficiently smooth on $\mathcal{M}$, and satisfying the following singularity condition

$$\forall x \in \mathcal{M} \quad \forall z \in T_x \quad V(x)z = 0.$$

Definition 2. *A matrix function $V(x) \in \Sigma$ is called P-positive definite if*

$$\forall x \in M \quad \forall z \in \mathbb{R}^n \quad P_x z \neq 0 \quad \Rightarrow \quad (z, V(x)z) > 0.$$

In the space Σ, we will consider a cone

$$\mathcal{K} = \{V \in \Sigma \mid V(x) \textit{ is a non-negative definite matrix } \forall\ x \in \mathcal{M}\},$$

and a subset of its internal elements

$$\mathcal{K}_P = \{V \in \Sigma \mid V \textit{ is P-positive definite}\}.$$

APPROXIMATION OF THE QUASIPOTENTIAL FOR STOCHASTIC SYSTEMS

To study the influence of random disturbances on the dynamic system, we consider a system of Ito's stochastic differential equations

$$dx \ = \ f(x)\, dt \ + \ \varepsilon\sigma(x)dw(t). \tag{2}$$

Here, $w(t)$ is a standard n-dimensional Wiener process, $\sigma(x)$ is a sufficiently smooth $n \times n$-matrix-function, and ε is a scalar parameter of the noise intensity.

Being forced by stochastic disturbances, the random trajectories of system (2) leave the invariant manifold $\mathcal{M}$, and form some probability distribution around it. The Fokker-Planck-Kolmogorov (FPK) equation

$$\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}p) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(f_i p) = 0, \qquad a_{ij} = [\sigma\sigma^\top]_{ij}$$

gives a detailed description of the stationary probabilistic density $p(x, \varepsilon)$. However, a direct use of this equation is technically difficult, even in simple cases. In these circumstances, asymptotic methods and approximations based on the quasipotential approach are used [3, 4, 5, 6]. For weak noise, an asymptotics of the stationary probability density $\rho(x, \varepsilon)$ can be written in the following form:

$$p(x, \varepsilon) \approx K \cdot e^{-\frac{v(x)}{\varepsilon^2}},$$

where the function $v(x)$ is a quasipotential.

The quasipotential is connected with some variational problem of the minimization of the action functional governed by the appropriate Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial v}{\partial x}, \sigma(x)\sigma^\top(x)\frac{\partial v}{\partial x}\right) + \left(f(x), \frac{\partial v}{\partial x}\right) = 0 \tag{3}$$

with conditions

$$v|_{\mathcal{M}} = 0\ , \quad v|_{\mathrm{U}\setminus\mathcal{M}} > 0. \tag{4}$$

However, it is also difficult to solve this equation analytically. In what follows, we will use a local description of the quasipotential in a small neighborhood of the manifold $\mathcal{M}$.

Due to properties (4), the quadratic form

$$\varphi(x) = \frac{1}{2}(\Delta(x), \Psi(\gamma(x))\Delta(x))$$

is the first approximation of the quasipotential $v(x)$ near the manifold $\mathcal{M}$: $v(x) = \varphi(x) + O(\|\Delta(x)\|^3)$. This quadratic form is parameterized by the symmetric non-negative $n \times n$-matrix function $\Psi(\gamma) = \dfrac{\partial^2 v}{\partial x^2}(\gamma)$ defined on the manifold $\mathcal{M}$. Note, that for non-singular noise, $\Psi(x) \in \mathcal{K}_P$.

Consider a matrix function $\Phi(x) = \Psi^+(x)$, where the sign "+" means a pseudo-inversion. The approximation $\varphi(x) = \frac{1}{2}(\Delta(x), \Phi^+(\gamma(x))\Delta(x))$ of the quasipotential $v(x)$ allows us to represent the asymptotics of the stationary density in a Gaussian form:

$$\rho(x, \varepsilon) \approx K \exp\left(-\frac{(\Delta(x), \Phi^+(\gamma(x))\Delta(x))}{2\varepsilon^2}\right)$$

with the covariance matrix $\varepsilon^2\Phi(x)$. We call the matrix $\Phi(x)$ the stochastic sensitivity function. This matrix characterizes a dispersion of the random trajectories of system (2) in the subspace N_x of the manifold $\mathcal{M}$.

Let x be an arbitrary point of the manifold $\mathcal{M}$. Using the system (1) solution $x(t) = X(t, x)$ $(X(0, x) = x)$ starting from this point, one can get t-parametric description of the function $\Psi(x)$:

$$V(t) = \Psi(x(t)).$$

It holds that $V(t) \in \mathcal{K}_P^x \subset \mathcal{K}^x \subset \Sigma^x$ where

$$\begin{aligned}
\Sigma^x &= \{V(t) \mid V(t) = V(x(t)), \quad V(x) \in \Sigma, \ t \in \mathbb{R}^1\} \\
\mathcal{K}^x &= \{V(t) \mid V(t) = V(x(t)), \quad V(x) \in \mathcal{K}, \ t \in \mathbb{R}^1\} \\
\mathcal{K}_P^x &= \{V(t) \mid V(t) = V(x(t)), \quad V(x) \in \mathcal{K}_P, \ t \in \mathbb{R}^1\}.
\end{aligned}$$

By differentiating the Hamilton-Jacobi equation (3) and substituting $x = x(t)$, for $V(t) = \Psi(x(t)) = \dfrac{\partial^2 v}{\partial x^2}(x(t))$ we get the following matrix differential Bernoulli equation:

$$\dot{V} + F^\top(t)V + VF(t) + VS(t)V = 0, \tag{5}$$

where

$$F(t) = \frac{\partial f}{\partial x}(x(t)), \quad S(t) = \sigma(x(t))\sigma^\top(x(t)).$$

Along with Eq. (5), consider the corresponding linear equation

$$\dot{W} = F(t)W + WF^\top(t) + P(t)S(t)P(t), \tag{6}$$

where $W(t)$ and $P(t) = P_{x(t)}$ are matrix functions.

The following theorem holds.

Theorem 1. *If the matrix function $W(t) \in \mathcal{K}_P^x$ is a solution of the equation (6), then the function $V(t) = W^+(t) \in \mathcal{K}_P^x$ is a solution of Eq. (5). If the matrix function $V(t) \in \mathcal{K}_P^x$ is a solution of Eq. (5), then the function $W(t) = V^+(t) \in \mathcal{K}_P^x$ is a solution of Eq. (6).*

Note that $W(t) = \Phi(x(t))$ characterizes the stochastic sensitivity of the manifold along the solution $x(t)$.

Eq. (6) introduced here with the help of the quasipotential, can be obtained in another way using the first approximation systems.

STOCHASTIC SENSITIVITY VIA FIRST APPROXIMATION SYSTEMS

Consider a deviation $x^\varepsilon(t) - x(t)$ of the forced solution $x^\varepsilon(t)$ of system (2) from the unforced solution $x(t)$ of the deterministic system (1).

For the asymptotics

$$z(t) = \lim_{\varepsilon \to 0} \frac{x^\varepsilon(t) - x(t)}{\varepsilon},$$

the following linear extension system can be written:

$$dx = f(x)\,dt, \qquad x \in \mathcal{M}, \tag{7}$$
$$dz = F(x)z dt + \sigma(x) dw(t), \qquad z \in \mathbb{R}^n, \quad F(x) = \frac{\partial f}{\partial x}(x).$$

In the deterministic case ($\sigma(x) = 0$), this system is

$$dx = f(x)\,dt, \quad x \in \mathcal{M}, \tag{8}$$
$$dz = F(x)z dt, \quad z \in \mathbb{R}^n.$$

The exponential stability of the manifold $\mathcal{M}$ in system (1) is equivalent [14] to the P-stability of the corresponding linear extension system (8).

In the stochastic case, the covariation matrix $Z(t) = \mathrm{cov}(z(t), z(t))$ satisfies the following system:

$$dx = f(x)\,dt, \tag{9}$$
$$dZ = (F(x)Z + ZF^{\top}(x) + S(x))\,dt.$$

Along with the system (7), consider a system

$$dx = f(x)\,dt, \tag{10}$$
$$dy = F(x)y dt + P_x \sigma(x) dw(t).$$

The covariation matrix $V(t) = \mathrm{cov}(y(t), y(t))$ of its solution $y(t)$ is governed by the system

$$\dot{x} = f(x), \tag{11}$$
$$\dot{V} = F(x)V + VF^{\top}(x) + P_x S(x) P_x.$$

Let x be an arbitrary fixed point of the manifold $\mathcal{M}$. The solution $x(t) = X(t, x) \quad (X(0, x) = x)$ of the deterministic system (1) gives us the following t-parametric representation:

$$F(t) = F(x(t)),\ G(t) = \sigma(x(t)),\ P(t) = P_{x(t)},\ S(t) = S(x(t)).$$

In this case, elements of the space Σ^x, the cone $\mathcal{K}^x$, and the set $\mathcal{K}_P^x$ are the matrix functions $V(t) = V(x(t))$.

Consider the operator $\mathcal{A}$ on the Σ^x defined as

$$\mathcal{A}[V] = \dot{V} + F^{\top}(t)V + VF(t).$$

For the scalar product of the elements $V, W \in \Sigma^x$ defined by

$$< V, W > = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{tr}(V(t)W(t))dt,$$

the operator

$$\mathcal{A}^*[W] = -\dot{W} + F(t)W + WF^{\top}(t)$$

is conjugate to $\mathcal{A}$. Using these operators, one can prove the following theorem.

Theorem 2. *Let the deterministic linear extension system (8) be P-stable. Then for any $x \in \mathcal{M}$, the following sentences are true:*

(a) *The matrix equation (6) has in the space Σ^x a unique solution $W(t) \in \mathcal{K}^x$. If $S(t) \in \mathcal{K}_P^x$, then $W(t) \in \mathcal{K}_P^x$;*

(b) *The system (10) has the solution $\bar{y}(t)$ with the covariance matrix* $\mathrm{cov}(\bar{y}(t), \bar{y}(t)) = W(t)$;

(c) *For any solution $y(t)$ of the system (10), the projection $P(t)V(t)P(t)$ of the covariance matrix $V(t)$ =* $\mathrm{cov}(y(t), y(t))$ *converges to the matrix $W(t)$:*

$$\lim_{t \to +\infty} (P(t)V(t)P(t) - W(t)) = 0;$$

(d) *For any solution* $y(t)$ *of the system (10), the projection* $P(t)y(t)$ *converges in the mean square to* $\bar{y}(t)$:

$$\lim_{t\to+\infty} \mathrm{E}||P(t)y(t) - \bar{y}(t)||^2 = 0;$$

(e) *For any solution* $z(t)$ *of the system (7), the projection* $P(t)Z(t)P(t)$ *of the covariance matrix* $Z(t) = \mathrm{cov}(z(t), z(t))$ *converges to the matrix* $W(t)$:

$$\lim_{t\to+\infty} (P(t)Z(t)P(t) - W(t)) = 0;$$

(f) *For any solution* $z(t)$ *of the system (7), the projection* $P(t)z(t)$ *converges in the mean square to* $\bar{y}(t)$:

$$\lim_{t\to+\infty} \mathrm{E}||P(t)z(t) - \bar{y}(t)||^2 = 0.$$

EXAMPLES

Consider now how this general theory can be applied to the concrete types of manifolds.

Stochastic sensitivity of the equilibrium. Let $\mathcal{M}$ be a stable equilibrium $\bar{x}$ of system (1). Then the stochastic sensitivity matrix W is constant. This matrix is a unique solution of the linear algebraic equation:

$$FW + WF^\top = -S, \qquad F = \frac{\partial f}{\partial x}(\bar{x}), \quad S = GG^\top, \quad G = \sigma(\bar{x}).$$

Stochastic sensitivity of the limit cycle. Let an invariant manifold $\mathcal{M}$ be a limit cycle corresponding to T-periodic solution $\bar{x}(t)$ of the system (1). The stochastic sensitivity matrix $W(t)$ is T-periodic. This matrix is a unique solution of the following boundary problem:

$$\begin{aligned} \dot{W} &= F(t)W + WF^\top(t) + P(t)S(t)P(t), \\ W(t+T) &= W(t), \qquad W(t)r(t) = 0. \end{aligned}$$

Here,

$$r(t) = f(\bar{x}(t)), \quad F(t) = \frac{\partial f}{\partial x}(\bar{x}(t)), \quad S(t) = G(t)G^\top(t), \quad G(t) = \sigma(\bar{x}(t)), \quad P(t) = P_{\bar{x}(t)}.$$

Details of the analysis of the stochastic sensitivity for toroidal manifolds can be found in [10].

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